WATER RESOURCES RESEARCH, VOL. 27, NO. 8, PAGE 2163, AUGUST 1991

Comment on "An Analytical Solution for One-Dimensional Transport in Heterogeneous Porous Media" by S. R. Yates

CONSTANTINOS V. CHRYSIKOPOULOS

Department of Civil Engineering, Stanford University, Stanford, California

The solution to the deterministic one-dimensional advection-dispersion equation with distance-dependent dispersion coefficient derived by *Yates* [1990] has many advantages due to its analytical nature. The proposed model is interesting with potential practical applications in laboratory heterogeneous packed column solute transport experiments and possibly in some field studies where the assumption of one-dimensional flow under constant velocity is valid. The author should be commended for the useful analytical results presented. The objective of this comment is to report a simpler solution than the one given by *Yates* [1990] for the case of zero initial concentration and constant flux boundary condition. The notation employed here is identical to *Yates* [1990].

The correct general Laplace time solution to the governing advection-dispersion model is given by *Yates* [1990, equation (9)], assuming that the division of $\bar{C}(\xi, s)$ by C_0 on the left-hand side is in error. In order to satisfy the downstream boundary condition, the Laplace time solution reduces to

$$\bar{C}(\xi, s) = A(s)\xi^{\gamma}K_{\gamma}[2\gamma(s+\beta)^{1/2}\xi]. \qquad (1)$$

The Laplace-transformed integration function, $A(s)$, is evaluated from the constant flux boundary condition [*Yates*, 1990, equation (15)]

$$-\frac{\xi_0}{2\gamma}\frac{d\bar{C}(\xi_0, s)}{d\xi} + \bar{C}(\xi_0, s) = \frac{C_0}{s}. \qquad (2)$$

Substituting (1) into (2) and taking the derivative yields

$$A(s) = \frac{C_0}{s(s+\beta)^{1/2}\xi_0^{\gamma+1}K_{\gamma+1}[2\gamma(s+\beta)^{1/2}\xi_0]}, \qquad (3)$$

where the following expression has been employed [*McLachlan*, 1955, p. 204; *Gradshteyn and Ryzhik*, 1980, p. 970]

$$z\frac{dK_{\nu}[z]}{dz} = \nu K_{\nu}[z] - zK_{\nu+1}[z]. \qquad (4)$$

Substituting (3) into (1) leads to

$$\frac{\bar{C}(\xi, s)}{C_0} = \left[\frac{\xi}{\xi_0}\right]^{\gamma}\frac{K_{\gamma}[2\gamma(s+\beta)^{1/2}\xi]}{s(s+\beta)^{1/2}\xi_0K_{\gamma+1}[2\gamma(s+\beta)^{1/2}\xi_0]}. \qquad (5)$$

The preceding equation may be used as an approximate solution in conjunction with numerical inversion of the Laplace transform by techniques such as the Stehfest algorithm [*Stehfest*, 1970], or Fourier series approximations [*Dubner and Abate*, 1968; *Crump*, 1976]. *Yates* [1990, equation (16)] is not in error, but (5) requires evaluation of only two modified Bessel functions and hence can be be considered computationaly less demanding. Following the procedure presented by *Yates* [1990, appendix], equation (5) is inverted from Laplace time variable s to real dimensionless time τ. The resulting solution is

$$\frac{C_f(\xi, \tau)}{C_0} = \left[\frac{\xi}{\xi_0}\right]^{\gamma}\left[\frac{K_{\gamma}[2\gamma\beta^{1/2}\xi]}{\beta^{1/2}\xi_0K_{\gamma+1}[2\gamma\beta^{1/2}\xi_0]} - \frac{2}{\pi}I_f\right], \qquad (6)$$

where

$$I_f = \int_{\beta^{1/2}}^{\infty}\frac{\exp[-\chi^2\tau]}{\chi} \cdot \left[\frac{\phi(\chi)Y_{\gamma}(\varepsilon)J_{\gamma+1}(\varepsilon_0) - \phi(\chi)J_{\gamma}(\varepsilon)Y_{\gamma+1}(\varepsilon_0)}{\phi(\chi)^2J_{\gamma+1}(\varepsilon_0)^2 + \phi(\chi)^2Y_{\gamma+1}(\varepsilon_0)^2}\right]d\chi. \qquad (7)$$

The solution (6) and (7) is more compact and easier to evaluate than that of *Yates* [1990, equations (17) and (18)]. It should also be noted that in (17) of *Yates* [1990] $\sqrt{b}$ should be replaced by $\sqrt{\beta}$.

REFERENCES

Crump, K. S., Numerical inversion of Laplace transforms using Fourier series approximation, *J. Assoc. Comput. Mach.*, *23*(1), 89–96, 1976.

Dubner, H., and J. Abate, Numerical inversion of Laplace transforms by relating them to finite Fourier cosine transform, *J. Assoc. Comput. Mach.*, *15*(1), 115–123, 1968.

Gradshteyn, I. S., and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic, San Diego, Calif., 1980.

McLachlan, N. W., *Bessel Functions for Engineers*, Oxford University Press, New York, 1955.

Stehfest, H., Algorithm 368 numerical inversion of Laplace transforms, *Commun. ACM*, *13*(1), 47–49, 1970.

Yates, S. R., An analytical solution for one-dimensional transport in heterogeneous porous media, *Water Resour. Res.*, *26*(10), 2331–2338, 1990.

C. V. Chrysikopoulos, Department of Civil Engineering, Stanford University, Stanford, CA 94305.

(Received March 29, 1991; accepted May 17, 1991.)

Paper number 91WR01389.
0043-1397/91/91WR-01389$02.00